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On Sum Criterion for the Existence of Some Iterative Methods

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Abstract

In this paper, we evoke an existence for two iterative methods of the system of linear equations. Our results agree with existing results and suggest a strong procedural condition for the convergence of the system of linear equations.

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1.0 Introduction

We consider the system of linear equation

$$Ax = b \tag{1}$$

where ,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},,$$

Problems concerning system of linear equations are often occurring in engineering and sciences[1-8]. Many researchers have developed several methods for solving the problems. Noor et al. [2] had used decomposition method for solving the system (1). Babolian et al.[6] have used Adomian decomposition method to derive an iterative method which is similar to the Jacobi iteration. Allahviranloo [4] used Adomian decomposition method for fuzzy system. Keramati [7] and Liu [8] have used homotopy perturbation method to derive some iterative methods for solving the system. Improving Newton-Raphson method for non linear equations by modified Adomian decomposition method was discussed in [3]. In this paper, we evoke a strong procedural condition for the decomposition method in [2] which is sufficient for the existence of system (1).

2.0 Preliminaries

We express system (1) as:

$$\left.\begin{cases} x_1 = (1 - a_{11})x_1 - a_{12}x_2 - \cdots - a_{in}x_n + b_1 \\ x_2 = -a_{21}x_1 + (1 - a_{22})x_2 - \cdots - a_{2n}x_n + b_2 \\ \vdots \\ x_n = -a_{n1}x_1 - a_{n2}x_2 - \cdots + (1 - a_{nn})x_n + b_n \end{cases}\right\} \tag{2}$$

If we set $\beta_{ij} = \delta_{ij} - a_{ij}$; $1 \le i,j \le n$, where δ_{ij} is the kronecker delta [].
Equation (2) can be written as

$$\begin{aligned} x1 &= \beta 11x1 - \beta 12x2 - \ldots - \beta 1nxn + b1 \\ x2 &= -\beta 21x1 + \beta 22x2 - \ldots - \beta 2nxn + b2 \\ xn &= -\beta n1x1 - \beta n2x2 - \ldots + \beta nnxn + bn \end{aligned} \tag{3}$$

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This implies (4)

$$x_i = \sum_{j=1}^{n} \beta_{ij} x_j + b_j, \quad i = 1,2,\dots,n$$

(5)

And in matrix form as

$x = Bx + b$

where $B = [\beta_{ij}]$, $1 \le i,j \le n$

Definition 2.1. *Let $(R^n, d(.,.))$ be a metric space. Let T be a mapping such that $T : R^n \longrightarrow R^n$. T is called a contraction on R^n if there is a positive real number $a < 1$ such that for all $x,y \in R^n$ $d(Tx,Ty) \le ad(x,y)$.*

The point $x \in R^n$ is fixed if it coincides with its image, i.e $x = Tx$. Now, define a mapping T on (5) which map R^n into itself, then (6)

$Tx = Bx + b$

and we can write (7)

$$Tx_i = \sum_{j=1}^{n} \beta_{ij} x_j + b_j, \quad i = 1,2,\dots,n.$$

3.0 Decomposed System

Let $h \neq 0$ be a parameter, for any splitting matrix Q and an auxiliary matrix H, we can decompose the system (1) as

$Qx + (hHA - Q)x = hHb$ (8)

Let $Q = D = diag(a_{ii})$, $i = 1,2,\dots$, then we can compare (8) with (5) if we let

$B^0 = -D^{-1}(hHA - D)$ and $b^0 = D^{-1}hHb$

so that (9)

$x(m+1) = B0x(m) + b0$

where m is a positive integer representing the number of iterations.

This is called the Adomian Jacobi method [5].

If we let $Q = -L + D$

where L is the lower triangular matrix with principal diagonal all zero.

From (8), we have

$$x^{(m+1)} = I - h(D-L)^{-1}HA)x^{(m)} + h(D-L)^{-1}Hb \quad (10)$$

$x^{(m+1)} \equiv B'' x^{(m)} + b''$

where $B'' = I - h(D-L)^{-1}HA)$ and $b^{00} = h(D-L)^{-1}Hb$

This is called the Adomian Gauss-Seidel method [5].

4.0 Main Results

In this section, we obtain the generalization of sum criterion and deduce some results which are sufficient for the existence of (1). The following Lemma is significant to our main results.

Lemma 3.1. *Let R^n be a metric space with metric d_p on R^n and T be a mapping of R^n into itself. Then, T is a contraction if*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |\beta_{ij}|^p < 1, \quad 1 \le p < \infty \quad (11)$$

Proof

Let R^n be a metric space, for $x,y \in R^n$, the metric d_p on R^n is defined by

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty \quad (12)$$

Let $T : R^n \longrightarrow R^n$ and $x,y \in R^n$. Using (6), we have;

$$d_p(Tx,Ty) = \left(\sum_{i=1}^{n} |Tx_i - Ty_i|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{n} \left|\sum_{j=1}^{n} \beta_{ij}(x_j - y_j)\right|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{i=1}^{n} \left|\sum_{j=1}^{n} \beta_{ij} \sum_{j=1}^{n}(x_j - y_j)\right|^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{n} \left|\sum_{j=1}^{n} \beta_{ij}\right|^p \left|\sum_{j=1}^{n}(x_j - y_j)\right|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |\beta_{ij}|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |x_j - y_j|^p\right)^{\frac{1}{p}} \text{(Schwarz's inequality)}$$

$$= \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |\beta_{ij}|^p\right)^{\frac{1}{p}} d_p(x,y)$$

Choose $\alpha = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |\beta_{ij}|^p\right)^{\frac{1}{p}}$ hence, $d_p(Tx,Ty) \leq \alpha d_p(x,y)$

Therefore, the mapping T on R^n is a contraction.

5.0 Remark

If the condition (11) is applied on either (9) or (10), then we obtain

n n
XX p p
|aij| <|aii|
i=1 j=1 (13)

Inequality (13) implies that the elements in the principal diagonal of matrix A must be large, and it will be called p-th sum criterion.

The condition (11) is sufficient for the convergence of (6) while (13) is sufficient for the convergence of (9) and (10) but the manner at which they converge differ depending on the choice of p. We consider the following cases

Case I: Suppose $p = 1$ in equation (13), we obtain

$$\sum_{i=1}^{n} |a_{ij}| < |a_{ii}|$$

for $j = 1,2,...,n$ and it is called column sum criterion

Case II: Suppose $p = 2$ in equation (13), we have the euclidean metric and the condition is called the square sum criterion given by

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 < a_{ii}^2$$

See [1] for similar work.

Case III: When $p = 3$ with the metric defined on R^n. For (13), we obtain the condition

n n
XX 3 3
|aij| <|aii| i=1 j=1

which is called the cubic sum criterion

Case IV: If $p = \infty$ with the metric d_∞ on R^n, we obtain the condition

$$\sup_i \sum_{j=1}^{n} |a_{ij}| < |a_{ii}|$$

This is called the row sum criterion.

We justify the aforementioned cases with the following examples:

Example 1:

Consider the system of linear equation

$$4x_1 - x_2 - x_4 = 0$$
$$-x_1 + 4x_2 - x_3 - x_5 = 5$$
$$-x_2 + 4x_3 - x_6 = 0$$
$$-x_1 + 4x_4 - x_5 = 6$$
$$-x_2 - x_4 + 4x_5 - x_6 = -2$$
$$-x_3 - x_5 + 4x_6 = 6$$

Solution

$$j = 1. \quad \sum_{i=1}^{6} |a_{ij}| = |a_{21}| + |a_{31}| + |a_{41}| + |a_{51}| + |a_{61}| = \frac{3}{4}$$

$$|a_{ii}| > \frac{3}{4} \text{ for } i = 1,2,3,4,5,6.$$

Case I is satisfied.

$$\sum_{i=1}^{6}\sum_{j=1}^{6} a_{ij}^2 = a_{12}^2 + a_{21}^2 + a_{23}^2 + a_{25}^2 + a_{32}^2 + a_{36}^2 + a_{41}^2 + a_{45}^2 + a_{52}^2 + a_{54}^2 + a_{56}^2 + a_{63}^2 + a_{64}^2 = \frac{7}{8}$$

$$|a_{ii}| > \frac{7}{8} \text{ for } i = 1,2,3,4,5,6.$$

Case II is satisfied

$$\sum_{i=1}^{6}\sum_{j=1}^{6} a_{ij}^3 = a_{12}^3 + a_{21}^3 + a_{23}^3 + a_{25}^3 + a_{32}^3 + a_{36}^3 + a_{41}^3 + a_{45}^3 + a_{52}^3 + a_{54}^3 + a_{56}^3 + a_{63}^3 + a_{64}^3 = \frac{7}{32}$$

$$|a_{ii}| > \frac{7}{32} \text{ for } i = 1,2,3,4,5,6.$$

Case III is satisfied

$$\sup_i \sum_{j=1}^{6} |a_{ij}| = \frac{1}{4} < 1$$

$$|a_{ii}| > \frac{1}{4} \text{ for } i = 1,2,3,4,5,6.$$

Case IV is satisfied

Example 2.

Consider the system

$2x_1 + x_2 + x_3 = 4$ $x_1 + 2x_2 + x_3 = 4$ $x_1 + x_2 + 2x_3 = 4$

Solution

Using the cases above, we have

$$\sum_{j=1}^{3} |a_{ij}| = 1 = |a_{ii}| \text{ for } i = 1,2,3.$$

This gives a non-expansive type.

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^2 = a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2 = \frac{6}{4}$$

$$|a_{ii}| < \frac{6}{4} \text{ for } i = 1,2,3.$$

Case II is not satisfied.

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^3 = a_{12}^3 + a_{13}^3 + a_{21}^3 + a_{23}^3 + a_{31}^3 + a_{32}^3 = \frac{3}{4}$$

$$|a_{ii}| > \frac{3}{4} \text{ for } i = 1,2,3.$$

Case III is satisfied.

$$\sup_i \sum_{j=1}^{3} |a_{ij}| = \frac{1}{2}$$

$|a_{ii}| > \frac{1}{2}$, for $i = 1,2,3$.

Case IV is also satisfied.

Example 3:

Consider the system

$x_1 + x_3 = 2 - x_1 + x_2 = 0$

$x_1 + 2x_2 - 3x_3 = 0$

Solution

$j = 1, \quad \sum_{i=1}^{3} |a_{ij}| = |a_{21}| + |a_{31}| = \frac{4}{3}$

$|a_{ii}| < \frac{4}{3}$, for $i = 1,2,3$.

Case I is not satisfied.

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^2 = a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2 = \frac{23}{9}$$

$|a_{ii}| < \frac{23}{9}$, for $i = 1,2,3$.

Case II is not satisfied

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}^3 = a_{12}^3 + a_{13}^3 + a_{21}^3 + a_{23}^3 + a_{31}^3 + a_{32}^3 = \frac{47}{27}$$

$|a_{ii}| < \frac{47}{27}$, for $i = 1,2,3$.

Case III is also not satisfied

$\sup_i \sum_{j=1}^{3} |a_{ij}| = 1$

$|a_{ii}| = 1$, for $i = 1,2,3$.

This is non-expansive.

6.0 Conclusion

It is generally known that if the system (5) of n linear equations satisfies at least one of the cases I, II, III and IV, then, it has precisely one solution. It would be observed that the example 1 is satisfied for all the cases, and hence, both the Jacobi and Gauss-seidel iterations converge. In example 2 , case I gives a non-expansive while case II is not satisfied, and therefore, the Jacobi iteration is not converge but Gauss-seidel iteration does. Only case IV gives a non-expansive in example 3, while cases I, II and III were not satisfied, so, Jacobi iteration converges while Gauss-seidel iteration diverges. On this note, the Jacobi and the Gauss-seidel iteration are not comparable, though the Gauss-seidel iteration converges faster than the Jacobi iteration but we are concerned on the criterion at which the iterative methods converge.

7.0 References

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