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NEW GENERALIZATIONS OF BAZILEVIČ MAPS

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Abstract. In this work we study, via Caratheodory maps with normalization by other than unity, a generalization of certain well-known subfamily of Bazilevič functions using the also well-known Salagean derivative operator.

1. Introduction

Let A be the class of analytic functions normalized by $f(0) = 0$ and $f'(0) = 1$ and having power series of the form $f(z) = z + a_2z^2 + \cdots$ and defined in the unit disk $E = \{z : |z| < 1\}$. Subsequent mention of $f(z)$ or z refers.

Let P be the class of Caratheodory functions $p(z) = 1 + c_1z + c_2z^2 + \ldots$ which are analytic in E and have positive real part in E, that is Re $p(z) > 0$.

Via a modified class of Caratheodory functions, P_λ, $\lambda = \eta + i\mu$ for real numbers $\eta > 0$ and μ, consisting of analytic functions $h(z) = 1 + i\mu/\eta + c_1z + c_2z^2 + \cdots$ normalized by $h(0) = 1 + i\mu/\eta$ and Re $h(z) > 0$, we studied in [5] Bazilevič maps defined in [6] by the integral

$$f(z) = \left\{ \frac{\alpha}{1+\beta^2} \int_0^z (p(t) - i\beta) t^{-\left(1+\frac{i\alpha\beta}{1+\beta^2}\right)} g(t)^{\frac{\alpha}{1+\beta^2}} dt \right\}^{\frac{1+i\beta}{\alpha}} \tag{1.1}$$

where $p \in P$ and $g(z) = z + b_2z^2 + \cdots$ is starlike, that is, Re $zg'(z)/g(z) > 0$; $\alpha > 0$ and β are real numbers and all powers mean principal values only.

In [5], we introduced:

DEFINITION 1. With all parameters having their preceding definitions, a function $f(z)$ belongs to the class $B(\lambda, g)$ if and only if

$$\frac{z\left(f(z)^\lambda\right)'}{\eta z^{i\mu} g(z)^\eta} \in P_\lambda.$$

The new definition includes the the Bazilevič maps as the case $\lambda = \alpha/(1 + i\beta)$, and some new hitherto elusive characterizations of those functions became known.

In the present paper, we propose a generalization using the Salagean derivative operator D^n, $n = 0, 1, 2, \cdots$, defined as

$$D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$$

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with $D^0 f(z) = f(z)$. The anti-derivative, I_n, of D^n is defined as

$$I_n = I(I_{n-1}f(z)) = \int_0^z \frac{I_{n-1}f(t)}{t}dt$$

with $I_0 f(z) = f(z)$. Both appeared in [9].

Then we say:

DEFINITION 2. With all parameters having their preceding definitions, a function $f(z)$ belongs to the class $B_n(\lambda, g)$ if and only if

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^{i\mu} g(z)^\eta} \in P_\lambda. \tag{1.2}$$

REMARK 1. For $n = 1$, we have the equivalent classes $B_1(\lambda, g) \equiv B(\lambda, g)$ investigated in [5].

REMARK 2. Using the anti-derivative, we have the integral representation

$$f(z) = \left\{\eta \lambda^{n-1} I_n \left[z^{i\mu} g(z)^\eta h(z)\right]\right\}^{\frac{1}{\lambda}} \tag{1.3}$$

for some $h \in P_\lambda$.

REMARK 3. In view of the fact that $B_n(\lambda, g)$ is well-defined with all parameters in line with the Bazilevic maps, then $B_1(\lambda, g)$ consists only of schlicht maps in the unit disk, as it is well known for purely real λ [10].

Then we shall investigate the particular case $g(z) = z$, which is hereby denoted $B_n(\lambda)$. That is:

DEFINITION 3. With all parameters having their preceding definitions, a function $f(z)$ belongs to the class $B_n(\lambda)$ if and only if

$$\frac{D^n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} \in P_\lambda. \tag{1.4}$$

The integral representation in this case is:

$$f(z) = \left\{\eta \lambda^{n-1} I_n [z^\lambda h(z)]\right\}^{\frac{1}{\lambda}} \tag{1.5}$$

for some $h \in P_\lambda$.

For this case, if $\mu = 0$, we have the class by $B_n(\eta)$ investigated in [1]. The new generalizations which present opportunities to unify many known results have also provided new insights into the study of the general Bazilevič maps of the form (1.1) as well as many new additions.

2. Preliminary lemmas

We shall require the following lemmas:

LEMMA 1. ([8]) *Let* $u = u_1 + u_2 i$ *and* $v = v_1 + v_2 i$. *Let* a *be a complex number with Re* $a > 0$ *and* $\psi(u,v)$ *a complex-valued function satisfying:*

(a) $\psi(u,v)$ *is continuous in a domain* Ω *of* $\mathbb{C}^2$,

(b) $(a,0) \in \Omega$ *and Re* $\psi(a,0) > 0$,

(c) *Re* $\psi(u_2 i, v_1) \leqslant 0$ *when* $(u_2 i, v_1) \in \Omega$ *and* $2v_1$ *Re* $a \leqslant -|a - u_2 i|^2$.

If $h = a + c_1 z + c_2 z^2 + \cdots$ *such that* $(h(z), zh'(z)) \in \Omega$ *and Re* $(h(z), zh'(z)) > 0$, *then Re* $h(z) > 0$.

It is known that the concept is non-vacous and that $\psi(u,v)$ defined by $\psi(u,v) = u + v/\lambda$, Re $\lambda > 0$ satisfies all the conditions of the lemma [8].

The next lemma removes the constraint $\zeta > 0$ in its versions in [3, 4]. The removability of the constraint is obvious in the proofs contained in both papers.

LEMMA 2. *For any complex number* ζ, *if either of* $D^{n+1}f(z)^{\zeta}/D^n f(z)^{\zeta}$ *or* $D^{n+1}f(z)/D^n f(z)$ *takes a value independent of* n, *then*

$$\frac{D^{n+1}f(z)^{\zeta}}{D^n f(z)^{\zeta}} = \zeta \frac{D^{n+1}f(z)}{D^n f(z)}.$$

LEMMA 3. ([5]) *Let* $h \in P_{\lambda}$. *Then*

$$Re \frac{zh'(z)}{h(z)} \geqslant \frac{-2r}{1-r^2}.$$

The next lemma is a consequence of the Caratheodory-Toeplitz inequality

$$\left| c_2 - \frac{c_1^2}{2} \right| \leqslant 2 - \frac{|c_1|^2}{2}$$

and was reported in [2].

LEMMA 4. ([2]) *If* $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$ *belongs to* P, *then for any complex constant* σ *we have the sharp inequality:*

$$\left| c_2 - \sigma \frac{c_1^2}{2} \right| \leqslant 2 \max\{1, |1-\sigma|\}.$$

3. Main results

First we prove:

THEOREM 1. $B_{n+1}(\lambda) \subset B_n(\lambda)$.

Proof. Define $h(z)$ by

$$\frac{D^n f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = h(z).$$

Then by some computation we find that

$$\frac{D^{n+1} f(z)^\lambda}{\eta\lambda^{n}z^\lambda} = h(z) + \frac{zh'(z)}{\lambda}$$

so that if $f \in B_{n+1}(\lambda)$, then

$$\mathrm{Re}\,\frac{D^{n+1} f(z)^\lambda}{\eta\lambda^{n}z^\lambda} = \mathrm{Re}\left(h(z) + \frac{zh'(z)}{\lambda}\right) > 0.$$

Now define $\psi(u,v) = u + v/\lambda$, $\mathrm{Re}\ \lambda > 0$. Noting that $a = 1 + i\mu/\eta$, then ψ satisfies all the conditions of Lemma 1, and thus follows that

$$\mathrm{Re}\,\frac{D^n f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = \mathrm{Re}\,h(z) > 0$$

meaning $f \in B_n(\lambda)$. □

COROLLARY 1. *The class $B_n(\lambda)$, $n \geqslant 1$, consists only of schlicht maps maps in the unit disk.*

Next we prove that functions of the class $B_n(\lambda)$ are σ-n-spiral, $|\sigma| \leqslant \pi/2$, in the sense that $\mathrm{Re}[e^{i\sigma}D^{n+1}f(z)/D^n f(z)] > 0$, in a subdisk $|z| < r_0(\eta)$. The concept of σ-spiral univalent functions (that is, $\mathrm{Re}[e^{i\sigma}zf'(z)/f(z)] > 0$) was introduced by Špaček in 1933 (see [7, 11]).

THEOREM 2. *Let $f \in B_n(\lambda)$. Then f is σ-n-spiral univalent in the disk $|z| < r_0(\eta)$ where $r_0(\eta)$ is given by*

$$r_0(\eta) = \frac{1}{\eta}\left(\sqrt{1+\eta^2} - 1\right)$$

where $\sigma = \tan^{-1}\frac{\mu}{\eta}$.

Proof. Since $f \in B_n(\lambda)$, then by definition there exists $h \in P_\lambda$ such that

$$\frac{D^n f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = h(z).$$

Then by simple computation we find that

$$D^{n+1}f(z)^{\lambda} = \eta\lambda^{n-1}z^{\lambda}\left(zh'(z)+i\mu h(z)\right)$$

so that

$$\frac{D^{n+1}f(z)^{\lambda}}{D^{n}f(z)^{\lambda}} = \lambda + \frac{zh'(z)}{h(z)}.$$

Using Lemma 2, we have

$$\lambda\frac{D^{n+1}f(z)}{D^{n}f(z)} = \lambda + \frac{zh'(z)}{h(z)}.$$

Taking $\sigma = \tan^{-1}\frac{\mu}{\eta}$ and applying Lemma 3, we have

$$\begin{aligned}\operatorname{Re} e^{i\sigma}\frac{D^{n+1}f(z)}{D^{n}f(z)} &> \frac{\eta}{|\lambda|} - \frac{2r}{(1-r^{2})|\lambda|}\\ &= \frac{\eta-\eta r^{2}-2r}{(1-r^{2})|\lambda|}\\ &> 0\end{aligned}$$

provided $|z| < r_0(\eta)$ as defined. □

THEOREM 3. *The class $B_n(\lambda)$ is closed under the integral*

$$F(z)^{\lambda} = \frac{\lambda+c}{z^{c}}\int_0^z t^{c-1}f(t)^{\lambda}dt,\ \ \lambda = \eta + i\mu.$$

Proof. From the integral $F(z)^{\lambda}$, we find

$$z(F(z)^{\lambda})' + cF(z)^{\lambda} = (\lambda+c)f(z)^{\lambda}$$

so that

$$\frac{D^{n+1}F(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}} + c\frac{D^{n}F(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}} = (\lambda+c)\frac{D^{n}f(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}}.$$

Now define $h \in P_{\lambda}$ by

$$\frac{D^{n}F(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}} = h(z).$$

Then we find that

$$\frac{D^{n+1}F(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}} = \lambda h(z) + zh'(z)$$

so that

$$\frac{D^{n}f(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}} = h(z) + \frac{zh'(z)}{\lambda+c},$$

which implies that

$$\operatorname{Re}\,\psi(h(z),zh'(z)) = \operatorname{Re}\left(h(z)+\frac{zh'(z)}{\lambda+c}\right) > 0$$

since $f \in B_n(\lambda)$. Hence by Lemma 1, we have Re $h(z) > 0$ which completes the proof. $\square$

THEOREM 4. *Let $f \in B_n(\lambda)$. Then*

$$|a_2| \leqslant \frac{2\eta|\lambda|^{n-2}}{|\lambda+1|^n}$$

and

$$|a_3| \leqslant \frac{2\eta|\lambda|^{n-2}}{|\lambda+2|^n}\max\left\{1, \left|\frac{(\lambda+1)^{2n}+\eta(1-\lambda)\lambda^{n-2}(\lambda+2)^n}{(\lambda+1)^{2n}}\right|\right\}.$$

The bounds are best-possible. Equalities are attained by

$$\begin{aligned} f(z) &= \left\{\eta\lambda^{n-1}I_n\left[z^\lambda\left(\frac{1+z}{1-z}+i\frac{\mu}{\eta}\right)\right]\right\}^{\frac{1}{\lambda}} \\ &= z+\frac{2\eta\lambda^{n-2}}{(\lambda+1)^n}z^2+\frac{2\eta\lambda^{n-2}}{(\lambda+2)^n}\left\{\frac{(\lambda+1)^{2n}+\eta(1-\lambda)\lambda^{n-2}(\lambda+2)^n}{(\lambda+1)^{2n}}\right\}z^3+\cdots. \end{aligned}$$

Proof. Since $f \in B_n(\lambda)$, then there exists $h \in P_\lambda$ such that

$$\frac{D^n f(z)^\lambda}{\eta\lambda^{n-1}z^\lambda} = h(z) = 1+i\frac{\mu}{\eta}+c_1z+c_2z^2+c_3z^3+\cdots.$$

Then we have

$$D^n f(z)^\lambda = \lambda^n z^\lambda+\eta\lambda^{n-1}c_1z^{\lambda+1}+\eta\lambda^{n-1}c_2z^{\lambda+2}+\eta\lambda^{n-1}c_3z^{\lambda+3}\cdots.$$

Using the Salagean integral operator, we have

$$f(z)^\lambda = z^\lambda+\frac{\eta\lambda^{n-1}c_1}{(\lambda+1)^n}z^{\lambda+1}+\frac{\eta\lambda^{n-1}c_2}{(\lambda+2)^n}z^{\lambda+2}+\frac{\eta\lambda^{n-1}c_3}{(\lambda+3)^n}z^{\lambda+3}\cdots.$$

Given that $f(z) = z+a_2z^2+a_3z^3+a_4z^4+\cdots$, we have

$$\begin{aligned} f(z)^\lambda &= z^\lambda+\lambda a_2z^{\lambda+1}+\left(\lambda a_3+\frac{\lambda(\lambda-1)}{2}a_2^2\right)z^{\lambda+2} \\ &\quad+\left(\lambda a_4+\lambda(\lambda-1)a_2a_3+\frac{\lambda(\lambda-1)(\lambda-2)}{6}a_2^3\right)z^{\lambda+3}+\cdots. \end{aligned}$$

so that

$$a_2 = \frac{\eta\lambda^{n-2}c_1}{(\lambda+1)^n}$$

from which the bound on a_2 follows using Caratheodory inequality $|c_1| \leqslant 2$, and

$$\begin{aligned} a_3 &= \frac{\eta\lambda^{n-2}c_2}{(\lambda+2)^n} - \frac{(\lambda-1)\eta^2\lambda^{2(n-2)}}{(\lambda+1)^{2n}}\frac{c_1^2}{2} \\ &= \frac{\eta\lambda^{n-2}}{(\lambda+2)^n}\left[c_2 - \frac{(\lambda-1)\eta\lambda^{n-2}(\lambda+2)^n}{(\lambda+1)^{2n}}\frac{c_1^2}{2}\right]. \end{aligned}$$

Hence, using Lemma 4 with $\sigma = \frac{(\lambda-1)\eta\lambda^{n-2}(\lambda+2)^n}{(\lambda+1)^{2n}}$, we have the required bound on the third coefficient of these functions.

Choosing $h(z) = (1+z)/(1-z) + i\mu/\eta$ in (1.5) we find that equality is attained by the extremal function given. □

Finally we state the bounds on the Fekete-Szego functionals for the class $B_n(\lambda)$. We omit the proof since it follows in similar manner as for the bounds on a_3. Here is it:

THEOREM 5. *Let $f \in B_n(\lambda)$. Then*

$$|a_3 - \rho a_2^2| \leqslant \frac{2\eta|\lambda|^{n-2}}{|2+\lambda|^n}\max\left\{1, \left|\frac{(1+\lambda)^{2n} + \eta(1+2\rho-\lambda)\lambda^{n-2}(\lambda+2)^n}{(\lambda+1)^{2n}}\right|\right\}.$$

The bounds are best-possible. Equalities are attained also by

$$\begin{aligned} f(z) &= \left\{\eta\lambda^{n-1}I_n\left[z^\lambda\left(\frac{1+z}{1-z} + i\frac{\mu}{\eta}\right)\right]\right\}^{\frac{1}{\lambda}} \\ &= z + \frac{2\eta\lambda^{n-2}}{(\lambda+1)^n}z^2 + \frac{2\eta\lambda^{n-2}}{(\lambda+2)^n}\left\{\frac{(\lambda+1)^{2n} + \eta(1-\lambda)\lambda^{n-2}(\lambda+2)^n}{(\lambda+1)^{2n}}\right\}z^3 + \cdots. \end{aligned}$$

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