CONJUGACY CLASSES OF THE ORDER-PRESERVING AND ORDER-DECREASING PARTIAL ONE-TO-ONE TRANSFORMATION SEMIGROUPS

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Abstract
In this paper I_n is considered to be the partial one-to-one transformation semigroup on $X_n = \{1,2,\cdots,n\}$. The order-preserving and order-decreasing partial one-to-one transformation semigroup IC_n) is defined to be the subsemigroups of I_n. The nilpotent and idempotent elements of IC_n were then obtained from its conjugacy classes using path structure to be n+1 and $\frac{1}{192}n$ – $\frac{1}{16}n$ $\frac{29}{}n^2 - n$ 2 respectively. The index and period of each conjugacy classes were also found. More so, some properties of conjugacy classes were stated.

Keywords: Semigroup, Order-preserving and Order-decreasing Partial one-to-one Transformation Semigroup, Path Structure, Conjugacy, Nilpotent and Idempotent Elements

Introduction
A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation. A transformation of X is a function from X to itself. Transformation semigroups is one of the most fundamental mathematical objects that occurs in theoretical computer science; where properties of language depend on algebraic properties of various transformation semigroups related to them Garba(1994).

Let $X_n = \{1,2,\cdots,n\}$, then a (partial) transformation $\alpha : Dom\alpha \subseteq X_n \to Im\alpha \subseteq X_n$ is said to be full (or total), if Dom $\alpha = X_n$, otherwise it is called strictly partial. The set of all partial transformation on *n*-object form a semigroup under the usual composition of transformation. It is denoted by P_n when it is partial, T_n when it is full (or total) and I_n when it is partial one-to-one.

Let S be a semigroup, an element $x \in$ is said to be conjugate to $\in$ if there exist $, v \in$ such that x $v$and v. For the purpose of this work, the definitions of conjugacy embedded in the theorems in Lipscomb (1996).

Two permutations are conjugate if they have the same cycle structure. The definition of conjugacy in arbitrary semigroup seems not to be unique as stated in Kudryavtseva and Mazorchuk (2007) and Kudryavtseva and Mazorchuk (2009) where they compared three approaches of conjugacy on semigroups. Also, Dauns (1989) and Lallement (1979) define monoids for free semigroups.

Cycle structure determines conjugacy class and can also be used to determine even permutations. Since the advent of semigroup, it was quite necessary to have an extension of cycle notation in the semigroups of transformation. Over the last 50 years, some authors have independently introduced the use of notations for elements of the transformation semigroups. The notations "links" and "cycles" were introduced in Munn (1957) where the former represents the nilpotent part of a transformation and the latter the permutation part. For instance, for a transformation α $\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}$ in I , which can be written as (12) [345], where (12) is the cycle and [345] the link. The author in Lipscomb (1996); introduced the notations proper path and circuits where proper path is for the nilpotent path and circuit the permutation part. For the same transformation as above, we write (12) (345], where (12) is the circuit and (345] the proper path. Also, other notations are in [Gomes & Howie (1987); and Sullivan (1987)]. In this paper proper path and circuit notations were used. A transformation $\alpha \in I_n$ is said to be order-preserving if $(\forall\, x,\ \in Dom\,\alpha\ \ x \leq\ \Longrightarrow x\alpha \leq\ \ \alpha$ Its order is given as IO_n $\binom{2n}{n}$ [see, Garba (1994)]. A transformation $\alpha \in I_n$ is said to be order-decreasing if $(\forall\, x \in Dom\,\alpha\ \ x\alpha \leq x$. Its order was given as $|ID_n|$ $\ _{n\ 1}$ $\sum^{n}\ \ ^{n}$ where $_n$ is the n Bell's number [see, Borwein, et.al. (1989)]. Moreso if $\alpha \in I_n$ there exist positive integers r and with r such that α α , then the integer r is called the index and m $- r$ the period of α. The Order-Preserving and Order-Decreasing partial one-to-one transformation semigroup $(IC_n$ is given as IC_n $IO_n \cap ID_n$. Its order is $C_{n\ 1}$ where C_n is the n Catalan number [see, Ganyushkin and Mazorchuk (2009)].

The aim of this paper is to investigate the conjugacy classes in order-preserving and order-decreasing partial one-to-one transformation semigroups $(IC_n$, examine the idempotent and nilpotent elements in it and obtaining the period and index for each conjugacy classes.

Theorem 1.1 Lipscomb (1996): Let $x,\ \in I_n$. Then the following holds:
(i) x is conjugate to if and only if they have the same path decomposition.
(ii) x is nilpotent if and only if its path decomposition are proper paths only.
(iii) x is idempotent if and only if all the paths in its decomposition is of length one.

Main Results

Conjugacy Classes of $\mathcal{C}_n$

Conjugacy classes of $\mathcal{C}_n$ can be enumerated by arranging its elements by their heights or fix of its path decomposition. In this paper, the classes were arranged by the latter. Let $\alpha \in\ \mathcal{C}_n$ be a transformation, the height of α $|Im\,\alpha|$ and the fix of α $(\alpha$ $|F(\alpha\ |$ $|x \in\ _n : x\alpha\ \ x|$
The nilpotent and idempotent elements in conjugacy classes were marked as $*$ and + respectively. The index and period of each conjugacy class were found. The first few numbers of the sequences of the order of IC_n is 1, 2, 5, 14, 42, 132, 429, 1430, 4863,... (see, Online Encyclopaedia of Integer Sequences (OEIS) A000108).

Table 2.1:
For IC_1

Fix of α	Conjugacy classes	Index	Period
0	(1] + *	1	1
1	(1)+	1	1

For IC_2

Fix of α	Conjugacy classes	Index	Period
0	(1](2] *+	1	1
	(21] *	2	1
1	(1)(2] +	1	1
2	(1)(2) +	1	1

For IC

Fix of α	Conjugacy classes	Index	Period
0	(1](2](3] *+	1	1
	(21](3] *	2	1
	(321] *	3	1
			1
1	(1)(2](3] +	1	
	(1)(32]	2	1
2	(1)(2)(3] +	1	1
3	(1)(2)(3) +	1	1

For IC

Fix of α	Conjugacy classes	Index	Period
0	(1](2](3](4] *+	1	1
	(21](3](4] *	2	1
	(21](43] *	2	1
	(321](4] *	3	1
	(4321] *	4	1
1	(1)(2](3](4] +	1	1
	(1)(32](4]	2	1
	(1)(234]	3	1
2	(1)(2)(3](4] +	1	1
	(1)(2)(43]	2	1
3	(1)(2)(3)(4] +	1	1
4	(1)(2)(3)(4) +	1	1

Since each of the conjugacy classes is a monogenic subsemigroup, we determined its index and period. The Index and Period of each conjugacy class forms its isomorphism class. From the above tables the followings lemma were deduced:

Lemma 2.1.1: Let α be a transformation in IC_n and an element $\alpha \in IC_n$, the Isomorphism between the monogenic subsemigroup $\langle \alpha \rangle$ and IC_n is of period 1.

Proof: By contraposition, let φ be a proper path and μ a circuit. If μ $x_1, x_2, \dots, x$ is a circuit of length $m \geq 2$ for some $x \in Dom\alpha$, $x_1 \leq x_2$ and $\alpha x_1 \geq \alpha x_2$ which is not an element of IC_n. Thus period of $\alpha \in IC_n$ is 1.

Lemma 2.1.2: Let φ and μ be path decomposition in $\alpha \in IC_n$ where φ is a proper path and μ a circuit. Enumerating its conjugacy classes according to the fix of α arranges its elements into partitions of positive integer for each fix of any n.

Proof: From the above proof, $\mu \in IC_n$ is of length 1 and therefore every circuit of IC_n is a fix of α. In each fix of α, the total conjugacy classes is equal to the different partitions of the greatest length of φ since partitions is writing "n" as a sum of positive integers.

Idempotent and Nilpotent Elements In C_n

Theorem 2.2.1: Let $\alpha \in IC_n$ and $|\ _n|$ denote the set of all idempotent elements in conjugacy classes of IC_n. The idempotent conjugacy classes is given as $|\ _n| \quad n \quad 1$.

Proof: For each fix of α that is {0, 1, 2, ..., n,}, there exists at least one idempotent element. Also idempotent elements of a particular fix of α fall under a conjugacy class. Thus there are $n \quad 1$. P idempotent conjugacy classes.

Conjecture 2.2.2: Let $|N_n|$ denote the number of nilpotent elements of the conjugacy classes of IC_n when n is even. $|N_n| \quad \frac{1}{192}n \ - \frac{1}{16}n \quad \frac{29}{}n^2 - n \quad 2$

Theorem 2.2.3: Let $\alpha \in IC_n$, the number of nilpotent conjugacy classes equals number of partitions of positive integer.

Proof: Nilpotent elements do not possess circuits or fixed points. Partition of a positive integer is writing a positive integer as a sum of positive integers. Let $\varphi \quad 1, 2, \dots, m$ be a proper path in the conjugacy classes of $\alpha \in IC_n$. The greatest height of $\varphi \quad 1, 2, \dots, m$ is $m - 1$ and can likened to the positive integer m. The next height will be $\varphi \quad (1, 2, \dots, m - 1\ (m$ and likened to the sum of positive integers $(m - 1 \quad 1$. Going down the heights of $\varphi \quad 1, 2, \dots, m$ we get the partitions $(1, 2, \dots, m\ , (1, 2, \dots, m - 1\ (m\ , (1, 2, \dots, m - 2\ (m - 1\ (m\ , \dots, (1\ (2\ \dots\ m - 2\ \ m - 1\ \ m$ which is the partition of a positive integer m.

Conjugacy classes of IC_n have circuits of length not exceeding 1 which implies that all circuits in the path decomposition of IC_n are fixed points. This observation leads to the formation of the theorem of fixed points below.

Theorem 2.2.4: Let $|F_n|$ be the number of fixed points in the conjugacy classes of IC_n. Then $|F_n| \quad \sum^{n}\ _1(\quad N_n$, where N is the number of partitions of a positive integer k.

Proof: Let path decomposition in the conjugacy classes be $(\mu_1\ (\mu_2\ \dots\ \mu\ \ (\varphi_1\ (\varphi_2\ \dots\ \varphi$ where all circuits and proper paths are of length 1. For a fixed point in a given path decomposition, it occurs into the number of partitions of the number of proper paths in the same decomposition which is $n - 1$.Two fixed points will occur into the number of partitions of $n - 2$. Also by convention, number of the partitions for the integer 0 is 1.Thus the result.

Remark 2.2.5: Conjugacy classes can also be applied to the numerous transformation semigroups in Algebra as can be seen in [Ugbene & Makanjuola (2013) and (2012)].

Remark 2.2.6: The sum of the number of partitions of a positive integer is the sequence 1, 2, 4, 7, 12, 19, 30, 45, 67, 97, 139, 195,...where n = 0, 1, ... and known as A000070 on OEIS.

Conclusion

The properties and numbers observed in this work are not limited in IC_n. Other properties and numbers such as the conjugacy class size, the wreath products of the conjugacy classes e.t.c. The results obtained in this paper are expected to be beneficial in areas such as automata theory, computational theory and formal languages.

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