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Numerical Studies for Solving Fractional Integro-Differential Equations by using Least Squares Method and Bernstein Polynomials

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Abstract

In this paper, two numerical methods for solving fractional integro-differential equations are proposed. The fractional derivative is considered in the Caputo sense. The proposed methods are least squares method aid of Bernstein polynomials function as the basis. The proposed method reduces this type of equation into systems to the solution of system of linear algebraic equations. To demonstrate the accuracy and applicability of the presented methods some test examples are provided. Numerical results show that this approach is easy to implement and accurate when applied to fractional integro-differential equations. We show that the method is effective and has high convergenc rate.

Keywords: Bernstein polynomials; Numerical studies; Convergency rate; Gaussian elimination method

Introduction

The fractional calculus has a long history from 30 September 1695, when the derivative of order $\propto = \frac{1}{2}$ has been described by Leibniz [1-4]. The theory of derivatives and integrals of non-integer order goes back to Leibniz, Liouville, Gr¨unwald, Letnikov and Riemann. There are many interesting books about fractional calculus and fractional differential equations [5-6]. The use of fractional differentiation for the mathematical modeling of real world physical problems has been wide spread in recent years, e.g., the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties, etc. Derivatives of non-integer order are defined in different ways, e.g., Riemann–Liouville, Gr¨unwald–Letnikow, Caputo and Generalized Functions Approach [4]. In this work we focus attention on Caputo's definition which turns out to be more useful in real-life applications since it is coupled with initial conditions having a clear physical meaning.

Furthermore the use of numerical method for solving fractional integro-differential equations cannot be over emphasized, for, it is of great importance to Mathematician, Engineers and Physicists. In recent years, much attention has been given for the solutions of fractional differential and integro-differential equations [7]. Proposed an efficient method for solving systems of fractional integro-differential equations using adomian decomposition method (ADM). Munkhammar JD proposed a numerical solution of fractional integro-differential equations by collocation method [8]. He JH used the Adomian Decomposition Method to solve fractional Integro-differential equations. ADM requires the construction of Adomian polynomials which are somehow difficult to obtain [9]. Homotopy Perturbation and Homotopy Analysis methods were applied to solve initial value problems of fractional order by Lanczos C [10]. These authors decomposed the given problems into basically two parts using linear and nonlinear operators. The basic assumption was that the solutions of the problem could be expressed as series of polynomials. The truncated parts of these polynomials are then solved to get the approximate solutions of the problems [11]. Employed application of the fractional differential transform method (FDTM) to fractional-order integro-differential equations with nonlocal boundary conditions [11] also gives some application of nonlinear fractional differential equations and their approximation [12]. Presented numerical approximation of fractional integro-differential equations by an Iterative Decomposition Method (IDM). In the work, approximate solution of each problem is presented as a rapidly convergent series of easily computable terms [13] applied least square method for treating nonlinear fourth order integro-differential equations [14,16-18] applied an efficient method for solving fractional differential equations using Bernstine polynomials [15] applied least squares method and shifted Chebyshev polynomial for solving fractional integro-differential equations. In his work he used shifted Chebyshev polynomial of the first kind as basis function.

In this work, using the idea of Momani and Qaralleh [15], we proposed an alternative method, called Standard and perturbed least square methods by Bernstein polynomial as basis function [19-22].

In this work, we are concerned with the numerical solution of the following linear fractional integro-differential equation by standard and perturbed least square methods using Bernstein polynomial as basis function

$$D^{\hat{a}}u(x) = f(x) + \int_0^1 k(x,t)u(x)dt, \ o \le x, t \le 1, \quad (1)$$

With the following supplementary conditions:

$$u^{(i)}(0) = \delta_i, \ n-1 < \alpha \le n, \ n \in N \quad (2)$$

Where $D^\alpha u(x)$ indicates the α th Caputo fractional derivative of $u(x), f(x),$

$K(x, t)$ are given smooth functions, x and t are real variables varying [0,1] and $u(x)$ is the unknown function to be determined.

Some Relevant Basic Definitions

Definition 1

A real function$(x),$ $x>0$ [Mohammed (2004)], is said to be in the space $C_\mu, \mu \in R$ if there

Exists a real number $P > \mu$ such that $f(x)=x^p f(x)$, where $f_1(x) \in C[0,1]$.

Definition 2

A function (x), $x>0$ is said to be in the space C_i^m, $m \in N \bigcup \{0\}$, $iff^m \, \mu$

Definition 3

The left sided Riemann–Liouvill fractional integral operator of the order $\mu^3 \, 0$ of a function $f \in C_\mu$, $\mu^3 -1$, is defined as

$$j^\alpha f(x) = \frac{1}{\Gamma(x)} \int_0^1 \frac{f(x)}{(x-t)^{1-\alpha}} dt, \alpha > 0, x > 0, \tag{3}$$

$$j^\alpha f(x) = f(x) \tag{4}$$

Definition 4

Let $\in C_{-1}^m 1, m \in N \bigcup \{0\}$. Then the Caputo fractional derivate of $f(x)$ is defined as

$$D^\alpha f(x) = \begin{cases} j^{m-\alpha} f^{m,(x), m-1<\alpha \le m, m \in N,} \\ \dfrac{D^m f(x)}{Dx^m}, \alpha = m \end{cases} \tag{5}$$

Hence, we have the following properties:

(1) $j^\alpha j^\nu f = j^{\alpha+\nu} f, \alpha, \nu > 0, f \in C_\mu, \mu > 0$

$$j^\alpha x^\gamma = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0$$

$$j^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0^+) \frac{x^k}{k}, x > 0, m-1 < \alpha \le m$$

$D\, j^\alpha f(x) = f(x), x > 0, m-1 < \alpha \le m$

$DC = 0, C$ is the constant

$$\begin{cases} 0, \beta \in N_0, \beta < [\alpha], \\ D^\alpha x^\beta = \dfrac{\tilde{A}(\beta+1)}{\tilde{A}(\beta-\alpha+1) x^{\beta-\alpha}}, \beta \in N_0, \beta \ge [\alpha] \end{cases}$$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0, 1.2, \ldots\}$

Definition 5

An integral equation is an equation in which the unknown function $y(x)$ appears under an integral signs [4] and [5].

A standard integral equation $y(x)$ is of the form:

$$y(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t) y(t) \, dt \tag{6}$$

Where $g(x)$ and $y(x)$ are the limits of integration, λ is a constant parameter, and $K(x, t)$ is a function of two variable x and t called the kernel or the nucleolus of the integral equation. The function $y(x)$ that will be determined appears under the integral sign and also appears inside and outside the integral sign as well. It is to be noted that the limits of integration $g(x)$ and $h(x)$ may be both variables, constants or mixed.

Definition 6

An integro-differential equation is an equation in which the unknown function $y(x)$ appears under an integral sign and contain ordinary derivatives $y^{(n)}(x)$ as well. A standard integro-differential equation is of the form:

$$y^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t) y(t) \, dt \tag{7}$$

Integral equations and integro-differential equations are classified into distinct types according to limits of integration and the kernel K(x; t) are as prescribed before.

1. If the limits of the integration are fixed, then the integral equation is called a Fredholm integral equation and is of the form:

$$y(x) = f(x) + \lambda \int_a^b K(x,t) y(t) \, dt \tag{8}$$

2. If at least one limits is a variable, then the equation is called a Volterra integral equation and is given as:

$$y(x) = f(x) + \lambda \int_a^x K(x,t) y(t) \, dt \tag{9}$$

Definition 7

Bernstein basis polynomials: A Bernstein polynomial [8] of degree n is defined by

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} \; i = 0, 1, \ldots \ldots n, \tag{10}$$

where,

$$\binom{n}{i} = \frac{n!}{i!(n-1)!} \tag{11}$$

Often, for mathematical convenience, we see $B_{i,n}(x)=0$ if $i<0$ or $i>n$.

Definition 8

Bernstein polynomials: A linear combination Bernstein basis polynomials

$$u_n(x) = \sum_i^n a_i B_{i,n}(x) \tag{12}$$

is the Bernstein polynomial of degree n where $a_i, i = 0, 1, 2, \ldots \ldots$ are constants.

Definition 9

Shifted Chebyshev polynomial of the first kind denoted by $T_n(x)$ is denoted by the following [11]:

$$T^*_n(x) = \cos\{n \cos^{-1}(2x-1)\}; n \ge 0 \tag{13}$$

and the recurrence relation is given by

$$T^*_{n+1}(x) = 2(2x-1)T^*_n(x) - T^*_{n-1}(x); n = 1, 2, \ldots. \tag{14}$$

With the initial condition

$$T^*_0(x) = 1, T^*_1(x) = 2x-1 \tag{15}$$

Definition 10

In this work, we defined absolute error as:

Absolute Error $= |Y(x) - y_n(x)|; 0 \le x \le 1$ \tag{16}

Demonstration of the Proposed Methods

In this section, we demonstrated the two proposed methods mentioned above.

Standard Least Squares Method (SLM)

The standard least square method with Bernstein polynomials as basis function is applied to find the numerical solution of fractional Integro-differential equation given in equation (1). This method is based on approximating the unknown function $u(x)$ by assuming an approximation solution of the form defined in equation (12).

Thus, substituting equation (12) into equation (1), we obtained

$$D^{\alpha}(\sum_{i=0}^{n} a_i B_{i,n}(x)) - (f(x) + \int_0^1 k(x,t)(\sum_{i=0}^{n} a_i B_{i,n}(t)dt) = 0 \quad (17)$$

Hence, the residual equation is obtained as

$$R(a_0, a_1, \ldots, a_n) = D^{\alpha}(\sum_{i=0}^{n} a_i B_{i,n}(x)) - [f(x) + \int_0^1 k(x,t)(\sum_{i=0}^{n} a_i B_{i,n}(t)dt)] \quad (18)$$

Thus, we minimized equation (18) as

$$S(a_0, a_1, \ldots, a_n) = \int_0^1 \left[R(a_0, a_1, \ldots, a_n) \right]^2 w(x)dx \quad (19)$$

Where $w(x)$ is the positive weight function defined in the interval, [a,b], thus, equation (19) is given

$$S(a_0, a_1, \ldots, a_n) =$$
$$\int_0^1 \left\{ (\sum_{i=0}^{n} D^{\alpha} a_i B_{i,n}(x) - [(f(x) + \int_0^1 k(x,t)(\sum_{i=0}^{n} a_i B_{i,n}(t)dt] \right\}^2 w(x)dx \quad (20)$$

We obtained the values of $a_i(i \geq 0)$ by finding the minimum value of S as:

$$\frac{\partial S}{\partial a_i} = 0, i = 0.1, \ldots, n \quad (21)$$

A. Applying equation (21) into equation (20) for various values of $i(i \geq 0)$;

We obtained $(n+i)$ algebraic system of equations in $(n+i)$ unknown constants $a_i's$. The systems of equations are then solved by Gaussian elimination method. The results of the unknown constants obtained are then substituted back into the approximate solution given by equation (12) to get the required approximation for the appropriate order.

Perturbed Least Squares Method (PLM)

The basic idea of the method as conceived by [10], is the substitution of equation (12) into a slightly perturbed equation (1) to obtain

$$D^{\alpha} u(x) = f(x) + \int_0^1 k(x,t)u(x)dt + H_n(x), \quad o \leq x, t \leq 1, \quad (22)$$

Where, $H_n(x) = \tau_n T_n^*(x) \quad (23)$

And $T_n^*(x)$ is the shifted Chebyshev polynomials defined in equation (13) and τ_n is a free tau parameter to be determined along with $a_i(i \geq 0)$. Equations (12) and (23) are substituted into equation (22) to get

$$D^{\alpha}(\sum_{i=0}^{n} a_i B_{i,n}(x)) - (f(x) + \int_0^1 k(x,t)(\sum_{i=0}^{n} a_i B_{i,n}(t)dt) - \tau_n T_n^*(x) = 0 \quad (24)$$

Hence the residual equation is defined as

$$R(a_0, a_1, \ldots, a_n) = D^{\alpha}(\sum_{i=0}^{n} a_i B_{i,n}(x)) - \tau_n T_n^*(x) - [f(x) + \int_0^1 k(x,t)(\sum_{i=0}^{n} a_i B_{i,n}(t)dt)] \quad (25)$$

Thus we minimized equation (24) by denoting

$$S(a_0, a_1, \ldots, a_n \text{ and } \tau_n) = \int_0^1 \left[R(a_0, a_1, \ldots, a_n) \right]^2 w(x)dx \quad (26)$$

Where all the parameters involved are mention above, thus, equation (24) is given

$$S(a_0, a_1, \ldots, a_n) = \int_0^1 \left\{ (\sum_{i=0}^{n} D^{\alpha} a_i B_{i,n}(x) - \tau_n T_n^*(x) - [(f(x) + \int_0^1 k(x,t)(\sum_{i=0}^{n} a_i B_{i,n}(t)dt] \right\}^2 w(x)dx \quad (27)$$

We obtained the values of $a_i(i \geq 0)$ and τ_n by finding the minimum value of S as:

$$\frac{\partial S}{\partial a_i} = 0, i = 0.1, \ldots, n \quad (28)$$

and $\frac{\partial S}{\partial \hat{o}_n} = 0 \quad (29)$

Applying equation (29) and (28) into equation (27) for various values of $i(i \geq 0)$

To obtain $(n+2)$ algebraic system of equations in $(n+2)$ unknown constants $a_i's$. The systems of equations are then solved by Gaussian elimination method. The results of the unknown constants obtained and then substituted back into the approximate solution given by equation (12) to get the required approximation.

Remark 1: The convergence and stability of the method were discussed in Taiwo (1991) while the existence and uniqueness of solution have been proved by Adeniyi (1991).

Numerical Examples

In this section, we demonstrated the proposed methods discussed above on some examples.

Example 1: Consider the following fractional Integro-differential equation:

$$D^{1/2} u(x) = \frac{\left(\frac{8}{3}\right)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtu(t)dt, 0 \leq x \leq 1, \quad (30)$$

Subject to $u(0)=0$ with exact solution $B(x)=x^2-x \quad (31)$

We have solved the above problem for value of n=3 in order to compare the results obtained with the exact solution. Also graphical representations of the result are presented.

For case n=3.

Thus, the approximate solution given in equation (12) becomes

$$u_3(x) \cong \sum_{i=0}^{3} a_i B_{i,3}(x) \quad (32)$$

Hence expanding equation (32) further, we have

$$u_3(x) = a_0(1 - 2x + 3x^2) + a_1(3x - 6x^2 + 3x^3) + a_1(3x^2 - 3x^3) + a_3 x^3 \quad (33)$$

Substituting (33) into (31), we have

$$D^{\frac{1}{2}}\left(a_0(1 - 2x + 3x^2) + a_1(3x - 6x^2 + 3x^3) + a_1(3x^2 - 3x^3) + a_3 x^3\right) = \frac{\left(\frac{8}{3}\right)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xt\left(a_0(1 - 2t + 3t^2) + a_1(3t - 6t^2 + 3t^3) + a_1(3t^2 - 3t^3) + a_3 t^3\right)dt, \quad (34)$$

Applying the Caputo properties on equation (34), we have

$$-\left[\frac{3\Gamma(1+1)x^{1-\frac{1}{2}}}{\Gamma(1-\frac{1}{2}+1)}\right]a_0 + \left[\frac{3\Gamma(2+1)x^{2-\frac{1}{2}}}{\Gamma(2-\frac{1}{2}+1)}\right]a_0 - \left[\frac{\Gamma(3+1)x^{3-\frac{1}{2}}}{\Gamma(3-\frac{1}{2}+1)}\right]a_0 + \left[\frac{3\Gamma(1+1)x^{1-\frac{1}{2}}}{\Gamma(1-\frac{1}{2}+1)}\right]a_1 - \left[\frac{6\Gamma(2+1)x^{2-\frac{1}{2}}}{\Gamma(2-\frac{1}{2}+1)}\right]a_1 +$$
$$\left[\frac{3\Gamma(3+1)x^{3-\frac{1}{2}}}{\Gamma(3-\frac{1}{2}+1)}\right]a_1 + \left[\frac{3\Gamma(2+1)x^{2-\frac{1}{2}}}{\Gamma(2-\frac{1}{2}+1)}\right]a_2 - \left[\frac{3\Gamma(3+1)x^{3-\frac{1}{2}}}{\Gamma(3-\frac{1}{2}+1)}\right]a_2 + \left[\frac{\Gamma(3+1)x^{4-\frac{1}{2}}}{\Gamma(3-\frac{1}{2}+1)}\right]a_3 \quad (35)$$
$$-\left(+1.504202866x^{\frac{3}{2}} - 1.128152150x^{0.5} + 0.08333333333x\right) - \int_0^1 xt\begin{pmatrix} a_0(1 - 2t + 3t^2) + a_1(3t - 6t^2 + 3t^3) + \\ a_1(3t^2 - 3t^3) + a_3 t^3 \end{pmatrix}dt = 0,$$

Simplifying equation (35), we have

$$\left(\begin{array}{l} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668x^{2.5}a_0 + 3.385137501x^{0.5}a_1 + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 \end{array}\right) = 0 \quad (36)$$

Hence, the residual equation is defined as

$$R(a_0, a_1..a_3) =$$

$$\left(\begin{array}{l} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668x^{2.5}a_0 + 3.385137501x^{0.5}a_1 + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 \end{array}\right) \quad (37)$$

In order to minimize equation (37), let

$$S(a_0, a_1,..a_3) = \int_0^1 \left[R(a_0, a_1.., a_3) \right]^2 dx \quad (38)$$

$$S(a_0, a_1..a_3) = \int_0^1 \left(\begin{array}{l} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668x^{2.5}a_0 + 3.385137501x^{0.5}a_1 + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 \end{array}\right)^2 dx \quad (39)$$

Applying equation (21) and (20) on equation (39), we have the following equations

$$\frac{\partial S}{\partial a_0} = \int_0^1 \left(\begin{array}{l} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668x^{2.5}a_0 + 3.385137501x^{0.5}a_1 + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - \\ 0.05xa_0 - 0.15xa_2 - 0.2xa_3 \end{array}\right)$$
$$\left(\begin{array}{l} -3.385137501x^{0.5} + 4.513516668x^{1.5} - \\ 1.805406668x^{2.5} - 0.05x \end{array}\right) dx = 0 \quad (40)$$

$$\frac{\partial S}{\partial a_1} = \int_0^1 \left(\begin{array}{l} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668 + 3.385137501x^{0.5} + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - \\ 0.1xa_1 - 0.05xa_0 - 0.15xa_2 - 0.2xa_3 \end{array}\right)$$
$$\left(\begin{array}{l} 3.385137501x^{0.5} + 9.023033336x^{0.5} + \\ 5.416220004x^{2.5}a_1 - 0.1x \end{array}\right) dx = 0 \quad (41)$$

$$\frac{\partial S}{\partial a_1} = \int_0^1 \left(\begin{array}{l} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668 + 3.385137501x^{0.5} + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - \\ 0.1xa_1 - 0.05xa_0 - 0.15xa_2 - 0.2xa_3 \end{array}\right)$$
$$\left(\begin{array}{l} 4.513516668x^{1.5} + 5.416220004x^{2.5} \\ -0.15x \end{array}\right) dx = 0 \quad (42)$$

$$\frac{\partial S}{\partial a_1} = \int_0^1 \left(\begin{array}{l} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668 + 3.385137501x^{0.5} + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - \\ 0.1xa_1 - 0.05xa_0 - 0.15xa_2 - 0.2xa_3 \end{array}\right)$$
$$\left(1.805406668x^{2.5} - 0.2x\right) dx = 0 \quad (43)$$

Integrating equations (40 - 43) with respect to x over the interval [0,1], we have four algebraic linear equations in 4 unknown constants. These algebraic linear equations are as follows:

$$1.023551680a_0 - 0.1798737760a_1 - 0.2375240337a_2 - 0.3649803917a_3 - -0.1390965247 = 0 \quad (44)$$

$$-0.1798737760a_0 + 0.2368150383a_1 + 0:1014859295a_2 - 0:1310140572a_3 + 0.1127452210 = 0 \quad (45)$$

$$-0.2375240337a_0 + 0.1014859295a_1 + 0.1854268035a_2 - 0.06737457209a_3 + 0.09561773339 = 0 \quad (46)$$

$$-0.3649803917a_0 - 0.1310140572a_1 - 0:06737457209a_2 + 0.3961016134a_3 - 0.06612184710 = 0 \quad (47)$$

The values obtained solving equations (44 - 47) by Gaussian elimination method are as follows:

$$a_0 = -3.487888303 \times 10^{-5},$$

$$a_1 = -0.3333012074,$$

$$a_2 = -3.3333014828,$$

$$a_3 = -3.552432041 \times 10^{-5}$$

These values are then substituted into equation (33), after simplifying we have the approximate solution as

$$u_3(x) = -0.00003487888303 - 0.9997989856x + 0.999798159x^2 + 0.0000001808x^3 \quad (48)$$

Demonstration of Perturbed Least Square Method

Example 1

Consider the following fractional Integro-differential equation:

$$D^{\frac{1}{2}}u(x) = \frac{\left(\frac{8}{3}\right)x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtu(t)dt, 0 \le x \le 1, \quad (49)$$

Subject to $u(0) = 0$, with exact solution $U(x) = x^2 - x$ $\quad (50)$

Solution

We have solved the problem above for the case n=3 with perturbation term $H_a(x)$ given in equation (23). Thus, equation (50) becomes

$$D^{\frac{1}{2}}u(x) = \frac{\left(\frac{8}{3}\right)x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xtu(t)dt + H_3(x), \quad (51)$$

Where $H_n(x) = \tau_n T_n^*(x)$ $\quad (52)$

and $H_3(x) = \tau_3 T_3^*(x) = H_n(x) = \tau_3(32x^3 - 48x^2 + 18x - 1)$ $\quad (53)$

The approximate solution given in equation (12) becomes

$$u_3(x) = \sum_i^3 a_i B_{i,3}(x) \quad (54)$$

Hence, expanding equation (54) further, we have

$$u_3(x) = a_0(1 - 2x + 3x) + a_1(3x - 6x^2 + 3x^3) + a_1(3x^2 - 3x^3) + a_3 x^3 \quad (55)$$

Substituting equations (53) and (55) into equation (51) we have

$$D^{\frac{1}{2}}\left(a_0(1 - 2x + 3x^2) + a_1(3x - 6x^2 + 3x^3) + a_1(3x^2 - 3x^3) + a_3 x^3\right) = \frac{\left(\frac{8}{3}\right)x^{\frac{3}{2}} - 2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \quad (56)$$
$$\int_0^1 xt\left(a_0(1 - 2t + 3t^2) + a_1(3t - 6t^2 + 3t^3) + a_1(3t^2 - 3t^3) + a_3 t^3\right) dt + \tau_3(32x^3 - 48x^2 + 18x - 1)$$

Applying the Caputo properties on equation (56) we have

$$-\left[\frac{3\Gamma(1+1)x^{1-\frac{1}{2}}}{\Gamma\left(1 - \frac{1}{2} + 1\right)}\right]a_0 + \left[\frac{3\Gamma(2+1)x^{2-\frac{1}{2}}}{\Gamma\left(2 - \frac{1}{2} + 1\right)}\right]a_0 - \left[\frac{\Gamma(3+1)x^{3-\frac{1}{2}}}{\Gamma\left(3 - \frac{1}{2} + 1\right)}\right]a_0 + \left[\frac{3\Gamma(1+1)x^{1-\frac{1}{2}}}{\Gamma\left(1 - \frac{1}{2} + 1\right)}\right]a_1 - \left[\frac{6\Gamma(2+1)x^{2-\frac{1}{2}}}{\Gamma\left(2 - \frac{1}{2} + 1\right)}\right]a_1 + \left[\frac{3\Gamma(3+1)x^{3-\frac{1}{2}}}{\Gamma\left(3 - \frac{1}{2} + 1\right)}\right]a_1 +$$
$$\left[\frac{3\Gamma(2+1)x^{2-\frac{1}{2}}}{\Gamma\left(2 - \frac{1}{2} + 1\right)}\right]a_2 - \left[\frac{3\Gamma(3+1)x^{3-\frac{1}{2}}}{\Gamma\left(3 - \frac{1}{2} + 1\right)}\right]a_2 + \left[\frac{\Gamma(3+1)x^{3-\frac{1}{2}}}{\Gamma\left(3 - \frac{1}{2} + 1\right)}\right]a_3 - \left[\frac{3\Gamma(3+1)x^{3-\frac{1}{2}}}{\Gamma\left(3 - \frac{1}{2} + 1\right)}\right]a_3 + \left[\frac{\Gamma(3+1)x^{3-\frac{1}{2}}}{\Gamma\left(3 - \frac{1}{2} + 1\right)}\right]a_3 \quad (57)$$
$$-\left(+1.504202866x^{\frac{3}{2}} - 1.128152150x^{0.5} + 0.08333333333\right) - \int_0^1 xt\left(a_0(1 - 2t + 3t^2) + a_1(3t - 6t^2 + 3t^3) + a_1(3t^2 - 3t^3) + a_3 t^3\right)$$
$$dt + \tau_3(32x^3 - 48x^2 + 18x - 1) = 0,$$

Simplifying equation (57), we have

$$\left(\begin{array}{l} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668x^{2.5}a_0 + 3.385137501x^{0.5}a_1 + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 \end{array}\right) - \tau_3(32x^3 - 48x^2 + 18x - 1) = 0 \quad (58)$$

Hence, the residual equation is defined as

$$R(a_0, a_1..a_3) =$$

$$\left(\begin{array}{l} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668x^{2.5}a_0 + 3.385137501x^{0.5}a_1 + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 - \tau_3(32x^3 - 48x^2 + 18x - 1) \end{array}\right) \quad (59)$$

In order to minimize equation (59), let

$$S\left(a_0, a_1, .. a_3, \tau_3\right) = \int_0^1 \left[R\left(a_0, a_1 .., a_3\right)\right]^2 dx \quad (60)$$

$$S\left(a_0, a_1 .. a_3\right) = \int_0^1 \begin{bmatrix} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668x^{2.5}a_0 + 3.385137501x^{0.5}a_1 + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 - \tau_3\left(32x^3 - 48x^2 + 18x - 1\right) \end{bmatrix}^2 dx \quad (61)$$

Applying equation (28) and (29) on equation (61), we have the following equations :

$$\frac{\partial S}{\partial a_0} = \int_0^1 \begin{pmatrix} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668x^{2.5}a_0 + 3.385137501x^{0.5}a_1 + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 - \tau_3\left(32x^3 - 48x^2 + 18x - 1\right) \end{pmatrix}$$
$$\left(-3.385137501x^{0.5} + 4.513516668x^{1.5} - 1.805406668x^{2.5} - 0.05x\right) dx = 0 \quad (62)$$

$$\frac{\partial S}{\partial a_0} = \int_0^1 \begin{pmatrix} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668 + 3.385137501x^{0.5} + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 - \tau_3\left(32x^3 - 48x^2 + 18x - 1\right) \end{pmatrix}$$
$$\begin{pmatrix} 3.385137501x^{0.5} + 9.023033336x^{0.5} + \\ 5.416220004x^{2.5}a_1 - 0.1x \end{pmatrix} dx = 0 \quad (63)$$

$$\frac{\partial S}{\partial a_2} = \int_0^1 \begin{pmatrix} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668 + 3.385137501x^{0.5} + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 - \tau_3\left(32x^3 - 48x^2 + 18x - 1\right) \end{pmatrix}$$
$$\left(4.513516668x^{1.5} + 5.416220004x^{2.5} - 0.15x\right) dx = 0 \quad (64)$$

$$\frac{\partial S}{\partial a_3} = \int_0^1 \begin{pmatrix} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668 + 3.385137501x^{0.5} + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 - \tau_3\left(32x^3 - 48x^2 + 18x - 1\right) \end{pmatrix}$$
$$\left(1.805406668x^{2.5} - 0.2x\right) dx = 0 \quad (65)$$

$$\frac{\partial S}{\partial \tau_3} = \int_0^1 \begin{pmatrix} -3.385137501x^{0.5}a_0 + 4.513516668x^{1.5}a_0 - 1.805406668 + 3.385137501x^{0.5} + \\ 9.023033336x^{0.5}a_1 + 5.416220004x^{2.5}a_1 + 4.513516668x^{1.5}a_2 + 5.416220004x^{2.5}a_2 + \\ 1.805406668x^{2.5}a_3 - 1.504202866x^{\frac{3}{2}} + 1.128152150x^{0.5} - 0.08333333333x - 0.1xa_1 - 0.05xa_0 - \\ 0.15xa_2 - 0.2xa_3 - \tau_3\left(32x^3 - 48x^2 + 18x - 1\right) \end{pmatrix}$$
$$\left(32x^3 - 48x^2 + 18x - 1\right) dx = 0 \quad (66)$$

Integrating the equations (62-66) with respect to x over the interval [0,1], we have we algebraic linear equations in 5 unknown constants. These algebraic linear equations are as follows:

$$1.023551680a_0 - 0.1798737760a_1 - 0.2375240337a_2 - 0.3649803917a_3 + 0.07300583245\ \tau_3 - 0.1390965247 = 0 \quad (67)$$

$$-0.1798737760a_0 + 0.2368150383a_1 + 0:1014859295a_2 - 0:1310140572a_3 + -0.2401096907\tau_3 + 0.1127452210 = 0 \quad (68)$$

$$-0.2375240337a_0 + 0:1014859295a_1 + 0.1854268035a_2 - 0.06737457209a_3 - 0.02522041758 + 0.09561773339 = 0 \quad (69)$$

$$-0.3649803917a_0 - 0.1310140572a_1 - 0:06737457209a_1 + 0.3961016134a_3 + 0.1423242758\tau_3 - 0.06612184710 = 0 \quad (70)$$

$$0.07300583245a_0 + 0.2401096907a_1 - 0.02522041758a_2 - 0.1423242758a_3 + 0.4857142857\tau_3 - 0.08842725214\ = 0 \quad (71)$$

The values obtained solving equations (67-71) by Gaussian elimination method, are as follows:

$a_0 = -1.080032206 \times 10^{-4}$,

$a_1 = -0.3333767679$,

$a_2 = -3.3333930138$,

$a_3 = -1.436727769 \times 10^{-4}$

$\tau_3 = 5.795092768 \times 10^{-7}$

The values are then substituted into equation (55), after simplification we the approximate solution as

$$u_3(x) = -0.0001080032206 - 0.9998062943x + 0.999757556x^2 + 0.000013067x^3 \quad (72)$$

Tables of Results

Numerical results of Example 1 (Table 1)

Examples 2

Consider the following fractional Integro-differential equation:

$$D^{\frac{5}{6}}u(x) = -\frac{3}{91}\frac{x^{\frac{1}{6}}\Gamma5/6\left(-91 + 216x^2\right)}{\pi} + (5 - 2e)x + \int_0^1 xe^t u(t) dt, \le x \le 1, \quad (73)$$

Subject to u(0)=0 with the exact equation U(x)=x−x³ (74)

Numerical results of Example 2 (Table 2)

## Graphical Representation of the Two Methods (Figures 1-4)

Conclusion

In this paper, least square method with the aid of Bernstein polynomials was successfully deduced for solving fractional integro-differential equations. The numerical results in the tables and graph show that the present method provides highly accurate numerical Solutions for solving these types of equations.

X	Exact Solution	Approximate Solution of standard least squares method (SLM)	Approximate solution of perturbed (PLM)	Absolute error of standard least squares method (SLM)	Absolute error of perturbed least squares method (PLM)
0.0	0.00	-0.00000348788	-0.00001080032	3.4878E-5	1.0800E-4
0.1	-0.09	-0.09001679563	-0.09009104407	1.6795E-5	9.1044E-5
0.2	-0.16	-0.16000274820	-0.16007885540	2.7482E-6	7.8855E-5
0.3	-0.21	-0.20999273540	-0.21007135870	7.2646E-6	7.1358E-5
0.4	-0.24	-0.23998675610	-0.24006847560	1.3243E-5	6.8475E-5
0.5	-0.25	-0.24998480930	-0.25007012800	1.5190E-5	7.0128E-5
0.6	-0.24	-0.23998689400	-0.24007623710	1.3106E-5	7.6237E-5
0.7	-0.25	-0.20999300890	-0.21008672480	6.9911E-6	8.6724E-5
0.8	-0.16	-0.16000315300	-0.16010151250	3.1530E-6	1.0151E-4
0.9	-0.09	-0.09001732530	-0.09012052186	1.7325E-5	1.2052E-4
1.0	0.00	-0.00000355247	-0.00000143674	3.5524E-5	1.4367E-4

Table 1: Numerical results of Example 1.

X	Exact Solution	Approximate Solution of standard least squares method (SLM)	Approximate solution of perturbed (PLM)	Absolute error of standard least squares method (SLM)	Absolute error of perturbed least squares method (PLM)
0.0	0.00	0.00010284974	0.00010996755	1.0284E-4	1.0996E-4
0.1	0.099	0.09906303620	0.09907034368	6.3036E-5	0.7034E-5
0.2	0.192	0.19202565970	0.19203342150	2.5659E-5	3.3421E-5
0.3	0.273	0.27299313320	0.27300160370	6.8668E-6	1.6037E-6
0.4	0.336	0.33596787000	0.33597729340	3.2130E-5	2.2706E-5
0.5	0.375	0.37495228320	0.37496289360	4.7716E-5	3.7106E-5
0.6	0.384	0.38394878610	0.38396080680	5.1213E-5	3.9193E-5
0.7	0.357	0.35695979180	0.35697343630	4.0208E-5	2.6563E-5
0.8	0.288	0.28798771360	0.28800318470	1.2286E-5	3.1847E-6
0.9	0.171	0.17103496450	1.71052455200	3.4964E-5	5.2455E-5
1.0	0.00	0.00010395790	0.00012365050	1.0395E-5	1.2365E-4

Table 2: Numerical results of Example 2.

Figure 1: Numerical results of Example 1.

Figure 3: Error of Example 1.

Figure 2: Numerical results of Example 2.

Figure 4: Error of Example 2.
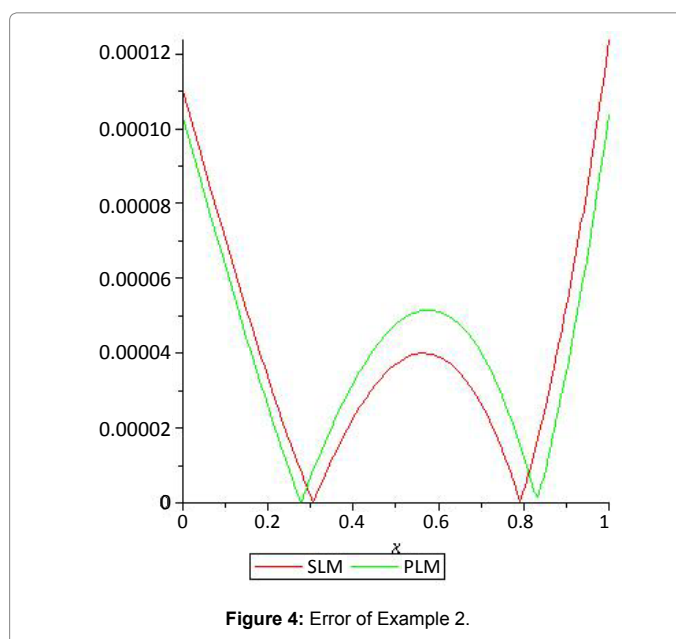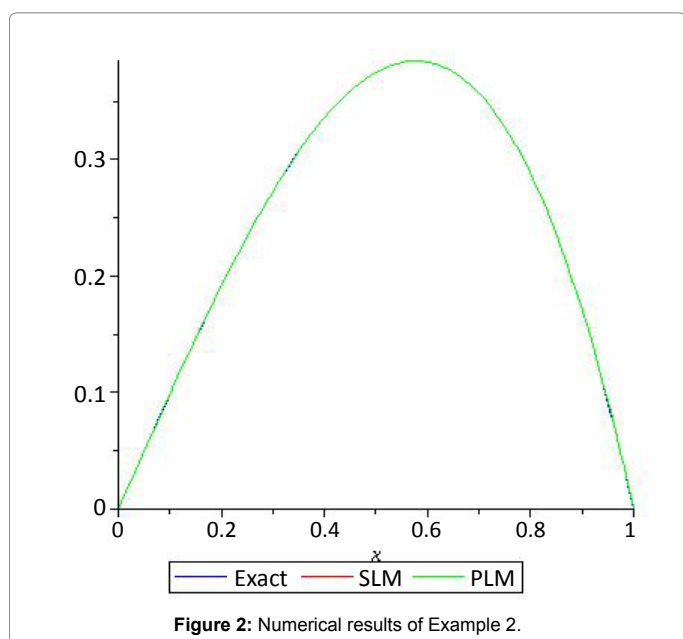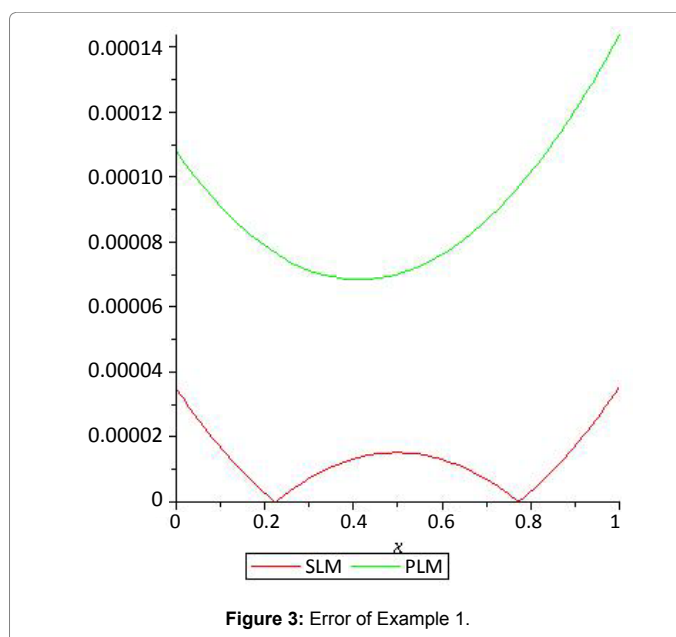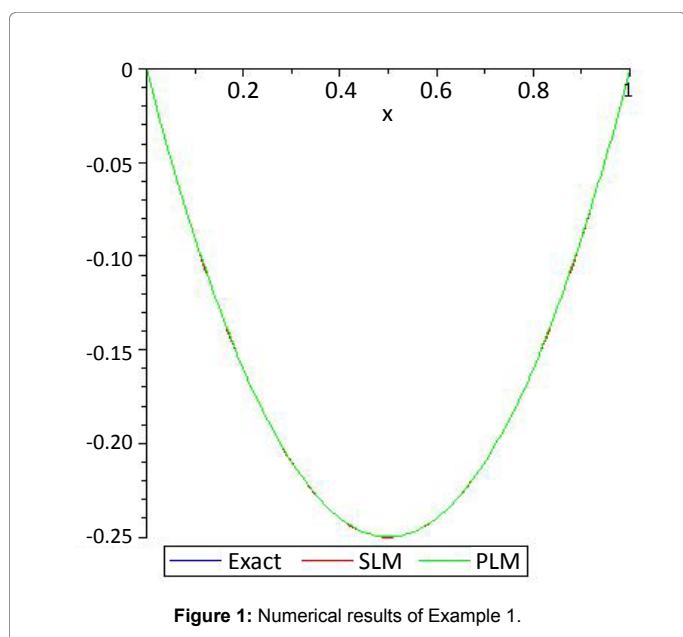
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